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STABILITY OF A COMPRESSED ELASTIC LAYER WEAKENED BY A CIRCULAR CRACK*

L.M. FILIPPOVA

A layer of non-linearly elastic incompressible material with a circular crack located at the middle of the layer parallel to its boundaries is considered. The layer is compressed by forces acting along the boundaries. Those values of the compression strain are sought for which buckling of the layer material occurs near the crack, i.e., opening of the crack occurs because of the elastic instability. The stability problem is reduced to a homogeneous Fredholm integral equation of the second kind with a continuous kernel dependent on the initial deformation parameter. Critical values of the compression are determined numerically as a function of the ratio between the layer thickness and the crack radius.

1. We assume that an elastic layer of incompressible isotropic material has a symmetrically located circular crack (slot) of radius a_0 in the undeformed state and experiences a finite strain due to a uniform axisymmetric load applied at infinity and acting in the plane of the crack. Since the crack is considered to be infinitely thin, its presence under such a loading is not felt and a homogeneous strain and a homogeneous stress field with the following components in cylindrical coordinates is realized in the layer:

$$\sigma_{rr} = \sigma_{\varphi\varphi} = \lambda \partial\Pi/\partial\lambda_1 - \lambda^{-2}\partial\Pi/\partial\lambda_3$$
$$\sigma_{zz} = \sigma_{r\varphi} = \sigma_{\varphi z} = \sigma_{rz} = 0. \tag{1.1}$$

Here λ_k $(k = 1, 2, 3)$ are the principal elongations (multiplicity of the elongation) $\Pi(\lambda_1, \lambda_2, \lambda_3)$ is a function of the specific strain potential energy of the incompressible material /1/, λ is the multiplicity of the elongation in the radial direction (for compression $0 < \lambda < 1$). Denoting the layer thickness in the undeformed configuration by $2a_0h_0$ and $a = \lambda a_0$ the crack radius in the initial deformed state, we find from the incompressibility condition that the thickness of the deformed layer will equal $2ah$, where $h = \lambda^{-3}h_0$.

A small deformation caused by loading the crack surface by an axisymmetrically distributed pressure $p(r)$ is superimposed on the finite deformation described.

We use the linearized equilibrium equations of a prestressed incompressible body under the axisymmetrical additional deformation /2/

$$\mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + \nu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial r\partial z}\right) + \frac{\partial q}{\partial r} = 0 \tag{1.2}$$
$$\nu\frac{\partial^2 u}{\partial r\partial z} + \varkappa\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial u}{\partial z} + \frac{\varkappa}{r}\frac{\partial w}{\partial r} + \frac{\partial q}{\partial z} = 0$$
$$\frac{\partial n}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$
$$\mu = 2\lambda^{-2}\Pi_3 + \lambda^2\Pi_{11} + \lambda^{-4}\Pi_{33} - 2\lambda^{-1}\Pi_{13}$$
$$\nu = \frac{\Pi_3 - \lambda^3\Pi_1}{\lambda^2 - \lambda^8}, \quad \varkappa = \frac{\lambda^4\Pi_3 - \lambda^7\Pi_1}{1 - \lambda^6}$$
$$\Pi_k = \frac{\partial\Pi}{\partial\lambda_k}, \quad \Pi_{ks} = \frac{\partial^2\Pi}{\partial\lambda_k\partial\lambda_s}.$$

Here r, z are dimensionless cylindrical coordinates (referred to the crack radius a in the initial deformed state), u, w are the radial and vertical components of the field of additional displacements, and q is the additional normal stress acting in horizontal sections of the layer.

The conditions expressing no load on the layer boundaries $z = \pm h$ have the form

$$\partial u/\partial z + \partial w/\partial r = 0, \; q' = 0. \tag{1.3}$$

By virtue of the symmetry of the problem relative to the middle plane of the layer, it is sufficient to examine half the layer $0 \leqslant z \leqslant h$ by setting the following mixed conditions at $z = 0$

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$$\partial u/\partial z + \partial w/\partial r = 0,\ 0 \leqslant r \leqslant \infty \tag{1.4}$$

$$q = -p(r),\ 0 \leqslant r < 1;\ w = 0,\ 1 < r < \infty . \tag{1.5}$$

Applying the Hankel integral transform, we construct the solution of Eqs.(1.2)

$$u(r,z) = \int_0^\infty (A_1\omega_1 s_1 + A_2\omega_1 c_1 + B_1\omega_2 s_2 + B_2\omega_2 c_2) J_1(\alpha r)\, d\alpha \tag{1.6}$$

$$w(r,z) = \int_0^\infty (A_1 c_1 + A_2 s_1 + B_1 c_2 + B_2 s_2) J_0(\alpha r)\, d\alpha$$

$$q(r,z) = \int_0^\infty [\omega_1\Delta_1(A_1 s_1 + A_2 c_1) + \omega_2\Delta_2(B_1 s_2 + B_2 c_2)]\, \alpha J_0(\alpha r)\, d\alpha$$

$$A_k = A_k(\alpha),\ B_k = B_k(\alpha),\ c_k = \operatorname{ch} \alpha\omega_k(h-z),$$
$$s_k = \operatorname{sh} \alpha\omega_k(h-z),\ k = 1, 2;\ \Delta_1 = \nu - \mu + \nu\omega_1^2,$$
$$\Delta_2 = \nu - \mu - \nu\omega_2^2 .$$

Here J_0 and J_1 are Bessel functions ω_1, ω_2 are roots of the equation $\nu\omega^4 - (\mu - 2\nu)\omega^2 + \varkappa = 0$, having a positive real part. Such roots exist if the elastic material satisfies the strict Hadamard inequality /2/.

Satisfying the boundary conditions (1.3) and (1.4), we obtain

$$B_1 = -\frac{1+\omega_1^2}{1+\omega_2^2} A_1,\quad B_2 = -\frac{\omega_1\Delta_1}{\omega_2\Delta_2} A_2 \tag{1.7}$$

$$A_2 = \frac{(1+\omega_1^2)\,\omega_2\Delta_2(\operatorname{ch}\alpha\omega_2 h - \operatorname{ch}\alpha\omega_1 h)\, A_1}{(1+\omega_2^2)(\omega_2\Delta_2 \operatorname{sh}\alpha\omega_1 h - \omega_1\Delta_1 \operatorname{sh}\alpha\omega_2 h)} .$$

The boundary conditions (1.5) result in dual integral equations in the function $A_1(\alpha)$ in terms of which the solution of the problem is expressed according to (1.6) and (1.7). This equation has quite an awkward form for a material of a general kind and it is not worth writing it down.

2. We will henceforth confine ourselves to the case of a neo-Hookean material /1/, for which the following expressions hold (G is the shear modulus)

$$\Pi = \tfrac{1}{2} G(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),\quad \nu = G\lambda^{-4} \tag{2.1}$$

$$\mu = G(\lambda^2 + 3\lambda^{-4}),\ \varkappa = G\lambda^2,\ \omega_1 = 1,\ \omega_2 = \lambda^3 .$$

After some manipulation and the introduction of a new unkown function $A(\alpha)$ the dual integral equation for this material takes the form

$$\int_0^\infty A(\alpha) J_0(\alpha r)\, d\alpha = 0,\quad r > 1 \tag{2.2}$$

$$\int_0^\infty A(\alpha)[C(\lambda) - g(\alpha,\lambda)]\, \alpha J_0(\alpha r)\, d\alpha = -\frac{ap(r)}{2G},\quad r < 1$$

$$C(\lambda) = \frac{(1+\lambda^6)^2 - 4\lambda^3}{1-\lambda^6},\quad g(\alpha,\lambda) = C(\lambda) - \frac{\Phi_2}{(1-\lambda^6)\Phi_1}$$

$$\Phi_1 = 4\lambda^3 \operatorname{sh} \alpha\lambda^{-3}h_0 \operatorname{ch} \alpha h_0 - (1+\lambda^6)^2 \operatorname{ch} \alpha\lambda^{-3}h_0 \operatorname{sh} \alpha h_0$$

$$\Phi_2 = 8\lambda^3(1+\lambda^6)^2(\operatorname{ch} \alpha\lambda^{-3}h_0 \operatorname{ch} \alpha h_0 - 1) - [16\lambda^6 + (1-\lambda^6)^4] \operatorname{sh} \alpha\lambda^{-3}h_0 \operatorname{sh} \alpha h_0 .$$

The radial component of the displacements in the layer is expressed in terms of the function $A(\alpha)$ as follows

$$u = \int_0^\infty \Big[\lambda^3(M_1 \operatorname{sh} \alpha\lambda^3(h-z) + M_2 \operatorname{ch} \alpha\lambda^3(h-z) + \frac{2\lambda^4}{(1-\lambda^6)\alpha}(N_1 \operatorname{ch} \alpha(h-z) + N_2 \operatorname{sh} \alpha(h-z))\Big] J_1(\alpha r) \frac{d\alpha}{\operatorname{sh} \alpha h}$$

$$M_1 = -\frac{4\lambda^4}{(1-\lambda^{12})\alpha} N_2,\quad M_2 = -\frac{(1+\lambda^6)\lambda}{(1-\lambda^6)\alpha} N_1$$

$$N_1 = 4\lambda^3(\operatorname{ch}\alpha h - \operatorname{ch}\alpha\lambda^3 h) A \Phi_1^{-1} \alpha \operatorname{sh}\alpha h$$

$$N_2 = A\Phi_1^{-1}[(1+\lambda^6)^2 \operatorname{sh}\alpha\lambda^3 h - 4\lambda^3 \operatorname{sh} \alpha h]\alpha \operatorname{sh}\alpha h .$$

Following the method described in /3/, we reduce the dual Eq.(2.2) to a Fredholm equation. We seek the unknown function in the form

$$A(\alpha)=\frac{1}{\alpha}\int_0^1 \varphi(t)(\cos\alpha t-\cos\alpha)\,dt. \tag{2.3}$$

The first relationship (for $r>1$) of the dual Eq.(2.2) is here satisfied identically for all continuously differentiable functions $\varphi(t)$ /3/.

Taking account of the known representations /4/

$$J_0(\alpha r)=\frac{2}{\pi}\int_0^{\pi/2}\cos(\alpha r\sin\theta)\,d\theta$$

$$\int_0^\infty J_0(\alpha r)\cos\alpha t\,d\alpha=\begin{cases}0, & r<t\\ (r^2-t^2)^{-1/2}, & r>t\end{cases}$$

and substituting (2.3) into the second relationship of the dual Eq.(2.2), and then setting

$$F(x)=C(\lambda)\varphi(x)-\frac{1}{\pi}\int_0^1\varphi(t)[K(t+x,\lambda)+K(t-x,\lambda)- \tag{2.4}$$
$$K(1+x,\lambda)-K(1-x,\lambda)]\,dt \qquad (0\leqslant x\leqslant 1)$$
$$K(x,\lambda)=\int_0^\infty g(\alpha,\lambda)\cos\alpha x\,d\alpha$$

we arrive at a Schloemilch integral equation

$$\int_0^{\pi/2}F(r\sin\theta)\,d\theta=-\frac{a}{2G}p(r)$$

whose solution is known /3/

$$F(x)=-\frac{a}{\pi G}\left[p(0)+x\int_0^{\pi/2}p'(x\sin\theta)\,d\theta\right]. \tag{2.5}$$

Substituting (2.5) into (2.4), we arrive at a Fredholm integral equation of the second kind in the function $\varphi(x)$. The problem of a crack in a prestressed layer can be reduced in an analogous manner based on (1.6) and (1.7) to a Fredholm equation even for other materials from the class of isotropic incompressible bodies.

3. Let us investigate the properties of the function $K(x,\lambda)$. It can be shown that the function $\Phi_1(\alpha,\lambda)$ has no real zeros except the point $\alpha=0$ for values $\lambda_*<\lambda<\infty$ where $\lambda_*=\gamma_*^{1/3}\approx 0.667$.

The set of real roots of the equation $\Phi_1(\alpha,\lambda)=0$ obviously consists of the root $\alpha=0$ and real roots of the equation

$$\operatorname{th}\alpha h_0/\operatorname{th}\alpha\lambda^{-3}h_0=4\lambda^3(1+\lambda^6)^{-2}. \tag{3.1}$$

We rewrite (3.1) as follows

$$\psi(y,\gamma)=f(\gamma),\quad \psi(y,\gamma)\equiv\frac{\operatorname{th}y\gamma}{\operatorname{th}y}\qquad f(\gamma)=\frac{4\gamma}{(1+\gamma^2)^2} \tag{3.2}$$
$$\gamma=\lambda^3>0,\ y=\alpha\lambda^{-3}h_0$$

It can be established that for all $0<y<\infty$

$$1\leqslant\psi(y,\gamma)\leqslant\gamma \text{ for } \gamma\geqslant 1 \tag{3.3}$$
$$\gamma\leqslant\psi(y,\gamma)\leqslant 1 \text{ for } \gamma\leqslant 1$$

The function $f(\gamma)-1$ has two zeros: 1 and $\gamma_*\approx 0.296$. Hence and from the expression for the derivative of this function it follows that $f(\gamma)>1$ for $\gamma_*<\gamma<1$ and $f(\gamma)<1$ for $\gamma<\gamma_*$ and $1<\gamma<\infty$. Taking (3.3) into account this means that (3.2) and, therefore, also (3.1) have no real roots for $\gamma_*<\gamma<\infty$.

Since the point $\alpha=0$ is not a pole of the function $g(\alpha,\lambda)$ this function has no poles on the real axis for $\lambda_*<\lambda<\infty$. Moreover, it can be established that the function $g(\alpha,\lambda)$ decreases exponentially as $\alpha\to\infty$. Therefore $K(x,\lambda)$ is a continuous function of x for $\lambda_*<\lambda<\infty$.

Note that the quantity λ_* is the critical value of the multiplicity of the elongation for an unbounded space with a crack. For $\lambda=\lambda_*$ instability of the unbounded body sets in (i.e., the layer of infinite thickness) weakened by a circular crack /2/.

Therefore, the problem of a crack under consideration is reduced to a Fredholm integral equation of the second kind with a continuous kernel. When there is no load on the crack surface $(p(r)=0)$, we arrive at a problem of determining the values of the parameter λ for which non-trivial solutions exist for the homogeneous Eq.((2.4) for $F(x)=0$). The minimum value of the parameter $\varepsilon=1-\lambda$, for which this equation has a non-zero solution is the critical deformation for which axisymmetric and symmetric buckling of the layer material relative to the $z=0$ plane occurs near the crack, i.e., opening of the crack occurs because of elastic instability. Below we give values of the critical deformation ε found by a numerical solution of the equations mentioned for a number of values of the relative half-thickness of the layer h_0

h_0	0.1	0.3	0.5	0.7	1
$\varepsilon\cdot 10^3$	7.22	50.9	111	168	228.

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GRAVITATIONAL ACCELERATION IN MINKOWSKI SPACE* **

L.I. SEDOV

The connections between models of the physical phenomenon of gravitation and geometrical models of space and time have been described previously /1-4/. As a result, an analysis was obtained of the macroscopic nature of gravitational interactions in the framework of four-dimensional pseudo-Riemannian spaces and three-dimensional Euclidean spaces in which there is Newtonian universal absolute time.

The corresponding theory is developed below for families K consisting of world lines associated with the free motion of individualized points that correspond to particles with a constant rest mass.

In the generalized coordinate system $\xi^1, \xi^2, \xi^3, \xi^4$ we find for individual points of the family K that $\xi^\alpha=\text{const}\ (\alpha=1,2,3)$ and ξ^4 is the time coordinate, changing along the world lines of K.

Generally speaking, for any family of associated world lines K, not necessarily for free motions of material particles in a pseudo-Riemannian space, we can introduce an associated canonical coordinate system $\xi^1, \xi^2, \xi^3, \tau$ where the metric has the following form at every point:

$$ds^2=c^2d\tau^2+2g_{\alpha 4}d\xi^\alpha d\tau+g_{\alpha\beta}d\xi^\alpha d\xi^\beta \quad (\alpha,\beta=1,2,3)$$

where the coordinate τ coincides with the proper global time on the world lines of K, and the components of the acceleration on the world lines are given by the formulae

$$g_{\alpha 4}=u_\alpha,\quad a_\alpha=\frac{\partial u_\alpha}{\partial\tau}=\partial g_{\alpha 4}(\xi^\alpha,\tau)/\partial\tau$$

where u_α is the covariant component of the four-dimensional velocity vector $\mathbf{u}$, directed along the tangent at each point of the corresponding world line K, and the contravariant components

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